Final Exam — Functional Analysis (WBMA033-05)

Tuesday 4 April 2023, 18.15–20.15h

University of Groningen

Instructions

1. The use of calculators, books, or notes is not allowed.
2. All answers need to be accompanied with an explanation or a calculation: only answering "yes", "no", or "42" is not sufficient.
3. If p is the number of marks then the exam grade is $G = 1 + p/10$.

Problem 1 (10 points)

Recall the following linear space from the lecture notes:

$$\ell^1 = \left\{ x = (x_1, x_2, x_3, \dots) \,:\, x_k \in \mathbb{K},\ \sum_{k=1}^{\infty} |x_k| < \infty \right\}.$$

We can equip this space with the following norms:

$$\|x\|_1 = \sum_{k=1}^{\infty} |x_k| \quad \text{and} \quad \|x\|_w = \sum_{k=1}^{\infty} e^{-k} |x_k|.$$

Show that these norms are *not* equivalent.

Problem 2 (10 + 5 + 5 + 10 = 30 points)

Consider the following linear operator:

$$T : \mathcal{C}([0,1], \mathbb{K}) \to \mathcal{C}([0,1], \mathbb{K}), \quad Tf(x) = f(x/2).$$

On the space $\mathcal{C}([0,1], \mathbb{K})$ we take the norm $\|f\|_\infty = \sup_{x \in [0,1]} |f(x)|$.

(a) Compute the operator norm of T.

(b) Show that every $\lambda \in (0,1]$ is an eigenvalue of T. Hint: take $f_p(x) = x^p$ with $p \geq 0$.

(c) Is T compact?

(d) Let $g(x) = x$ and $|\lambda| > \|T\|$. Explicitly compute all functions $f \in \mathcal{C}([0,1], \mathbb{K})$ such that $Tf - \lambda f = g$.

Problem 3 (10 + 10 = 20 points)

Let X be a Hilbert space over $\mathbb{K} = \mathbb{C}$ and let $T \in B(X)$. Prove the following statements:

(a) $\ker T^* = (\operatorname{ran} T)^\perp$;

(b) $\ker T^*T = \ker T$.

Turn page for problems 4 and 5!

Problem 4 (5 + 10 = 15 points)

Let X be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and let $S : X \to X$ be a linear operator such that

$$\langle Sx, y \rangle = -\langle x, Sy \rangle \quad \text{for all} \quad x, y \in X.$$

For any fixed $y \neq 0$ define the linear map $T_y : X \to \mathbb{K}$ by

$$T_y(x) = \frac{\langle Sx, y \rangle}{\|y\|}.$$

(a) Show that for all $x \in X$ we have $\sup\{|T_y(x)| : y \neq 0\} < \infty$.

(b) Prove that S is bounded by applying the Uniform Boundedness Principle.

Problem 5 (15 points)

Let X be a normed linear space and assume that $x, y \in X$ are distinct. Use the Hahn-Banach Theorem to prove that there exists a bounded linear map $f : X \to \mathbb{K}$ such that

$$\|f\| = 1 \quad \text{and} \quad f(x) \neq f(y).$$

Hint: consider the 1-dimensional space spanned by $z = x - y$.

End of test (90 points)

Solution of problem 1 (10 points)

If norms $\|\cdot\|_1$ and $\|\cdot\|_w$ are equivalent, then there exist constants $m, M > 0$ such that

$$m\|x\|_w \leq \|x\|_1 \leq M\|x\|_w \quad \text{for all} \quad x \in \ell^1.$$

(3 points)

For the vector $x^n = (0, \dots, 0, 1, 0, \dots)$, where the 1 is at the n-th position, we have

$$me^{-n} \leq 1 \leq Me^{-n} \quad \text{for all} \quad n \in \mathbb{N}.$$

(4 points)

Letting $n \to \infty$ gives the inequality $0 \leq 1 \leq 0$ which is clearly false. Therefore, the given norms $\|\cdot\|_1$ and $\|\cdot\|_w$ are not equivalent.
(3 points)

Note: there are other approaches. For example, for $x^n = (0, \dots, 0, e^n, 0, \dots)$ we would have $\|x^n\|_w = 1$ while $\|x^n\|_1 = e^n \to \infty$, which again contradicts the inequalities we are supposed to have for equivalent norms.

Solution of problem 2 (10 + 5 + 5 + 10 = 30 points)

(a) Let $f \in \mathcal{C}([0,1], \mathbb{K})$ be arbitrary and note that

$$\{f(x/2) : x \in [0,1]\} = \{f(x) : x \in [0,1/2]\} \subset \{f(x) : x \in [0,1]\}.$$

This gives

$$\|Tf\|_\infty = \sup_{x\in[0,1]} |f(x/2)| \leq \sup_{x\in[0,1]} |f(x)| = \|f\|_\infty.$$

(3 points)

We also conclude that for all nonzero $f \in \mathcal{C}([0,1], \mathbb{K})$ we have $\|Tf\|_\infty / \|f\|_\infty \leq 1$.
(3 points)

Finally, if we take any constant function f (i.e. $f(x) = c$ for all $x \in [0,1]$) we have that both $\|Tf\|_\infty = |c|$ and $\|f\|_\infty = |c|$ so that $\|Tf\|_\infty / \|f\|_\infty = 1$. We therefore conclude that

$$\|T\| = \sup_{f\in\mathcal{C}([0,1],\mathbb{K}), f\neq 0} \frac{\|Tf\|_\infty}{\|f\|_\infty} = 1.$$

(4 points)

(b) For $f_p(x) = x^p$ with $p \geq 0$ we have that

$$Tf_p(x) = \left(\frac{x}{2}\right)^p = \frac{x^p}{2^p} \quad \text{for all} \quad x \in [0,1],$$

which means that $Tf_p = \lambda_p f_p$ where $\lambda_p = 1/2^p$. Since $f_p \neq 0$ we have shown that λ_p is an eigenvalue of T.
(4 points)

Observe that the function $p \mapsto 1/2^p$ is a bijection from $[0, \infty)$ onto $(0, 1]$. Therefore, we have shown that all elements of the interval $(0, 1]$ are eigenvalues of T.
(1 point)

(c) If T were a compact operator, then T would have at most countably many eigenvalues. However, in part (b) we have shown that T has uncountably many eigenvalues. We conclude that T cannot be compact.
(5 points)

(d) There are at least two ways to solve this equation.

Method 1: using inversion by geometric series. Note that

$$T - \lambda = -\lambda\left(I - \frac{T}{\lambda}\right).$$

Since $|\lambda| > \|T\|$ we have $\|T/\lambda\| < 1$, and since $\mathcal{C}([0,1], \mathbb{K})$ is a Banach space we can use inversion by geometric series:

$$(T - \lambda)^{-1} = -\frac{1}{\lambda}\left(I - \frac{T}{\lambda}\right)^{-1} = -\frac{1}{\lambda}\sum_{k=0}^{\infty} \frac{T^k}{\lambda^k}.$$

(5 points)

The unique solution of the equation $Tf - \lambda f = g$ is given by

$$f = (T - \lambda)^{-1} g = -\frac{1}{\lambda} \sum_{k=0}^{\infty} \frac{1}{\lambda^k} T^k g.$$

Plugging in the function $g(x) = x$ gives

$$f(x) = -\frac{1}{\lambda} \sum_{k=0}^{\infty} \frac{1}{\lambda^k} \cdot \frac{x}{2^k} = -\frac{x}{\lambda(1 - 1/2\lambda)} = \frac{2x}{1 - 2\lambda}.$$

(5 points)

Method 2: ad hoc. Since $|\lambda| > \|T\|$ we know that $\lambda \in \rho(T)$ which means that $T - \lambda$ is invertible. In particular, the equation $Tf - \lambda f = g$ has precisely one solution.
(4 points)

As an educated guess, assume that the solution f is of the form $f(x) = ax$ for some constant a. Plugging this into the equation gives the equality

$$\frac{ax}{2} - \lambda ax = x \quad \text{for all} \quad x \in [0, 1].$$

This gives $a = 2/(1 - 2\lambda)$.
(6 points)

Solution of problem 3 (10 + 10 = 20 points)

(a) We have the following equivalences:

$$\begin{aligned} x \in \ker T^* &\Leftrightarrow T^* x = 0 \\ &\Leftrightarrow \langle T^* x, y \rangle = 0 \quad \text{for all} \quad y \in X \\ &\Leftrightarrow \langle x, Ty \rangle = 0 \quad \text{for all} \quad y \in X \\ &\Leftrightarrow x \in (\operatorname{ran} T)^\perp, \end{aligned}$$

which shows the desired equality.
(10 points; about 2 points per equivalence)

(b) We have the following equivalences:

$$\begin{aligned} x \in \ker T^*T &\Leftrightarrow T^*Tx = 0 \\ &\Leftrightarrow Tx \in \ker T^* \\ &\Leftrightarrow Tx \in (\operatorname{ran} T)^\perp \quad \text{by part (a)} \\ &\Leftrightarrow Tx = 0, \end{aligned}$$

where the last equivalence is due to the fact that for any subspace $V \subset X$ we have that $V \cap V^\perp = \{0\}$. This shows the desired equality.
(10 points; about 2 points per equivalence)

Solution of problem 4 (5 + 10 = 15 points)

(a) Let $x \in X$ be arbitrary. For all nonzero $y \in X$ the Cauchy-Schwarz inequality gives

$$|T_y(x)| = \frac{|\langle Sx, y\rangle|}{\|y\|} \leq \frac{\|Sx\|\,\|y\|}{\|y\|} = \|Sx\|.$$

(4 points)

This implies that $\sup\{|T_y(x)| \,:\, y \neq 0\} \leq \|Sx\| < \infty$.
(1 point)

(b) Since X is a Hilbert space (and therefore also a Banach space), part (a) implies that we can apply the Uniform Boundedness Principle. This gives the existence of a constant $c > 0$ such that

$$\sup_{y\neq 0} \|T_y\| \leq c.$$

(5 points)

Note that the given property of S also implies that

$$T_y(x) = -\frac{\langle x, Sy\rangle}{\|y\|}.$$

In particular, taking $x = Sy/\|y\|$ gives

$$\frac{\|Sy\|^2}{\|y\|^2} = |T_y(Sy/\|y\|)| \leq \|T_y\|\,\frac{\|Sy\|}{\|y\|}$$

(3 points)

Therefore, for all nonzero $y \in X$ we have

$$\frac{\|Sy\|}{\|y\|} \leq \|T_y\| \leq c$$

which means that S is a bounded operator.
(2 points)

Solution of problem 5 (15 points)

Since $x, y \in X$ are distinct the vector $z = x - y$ is nonzero and thus the space $V = \operatorname{span}\{z\}$ is 1-dimensional. Define the linear map

$$g : V \to \mathbb{K}, \quad g(\lambda z) = \lambda \|z\|.$$

Then we have

$$\|g\| = \sup_{v \in V, v \neq 0} \frac{|g(v)|}{\|v\|} = \sup_{\lambda \neq 0} \frac{|g(\lambda z)|}{\|\lambda z\|} = \sup_{\lambda \neq 0} \frac{|\lambda| \, |z\|}{|\lambda| \, \|z\|} = 1.$$

(10 points)

By the Hahn-Banach theorem there exists a map $f : X \to \mathbb{K}$ such that $f \restriction V = g$ and $\|f\| = \|g\| = 1$.
(3 points)

Note that we have

$$f(x - y) = g(x - y) = g(z) = \|z\| = \|x - y\| \neq 0,$$

This implies that $f(x) \neq f(y)$.
(2 points)